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On the Deformation of Time Harmonic Flows[1]

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Abstract

It is shown that time-harmonic motions of spherical and toroidal surfaces can be deformed non-locally without loosing the existence of infinitely many constants of the motion.

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As pointed out some time ago [1], surfaces moving through $\mathbf{R}^3$ in such a way that their (normal) velocity is always equal to the local surface-area density, $\sqrt{g}$ (divided by some non-dynamical 'reference-density', ρ), have the property that the time, at which the surface Σ_t reaches a point in space, is a harmonic function. These motions are related to certain reductions of self-dual $SU(N)$ Yang-Mills theories (which play a central role in the construction of monopoles, cp. [2]), and the Lax-pair formulation of the latter can be taken over identically ([3],[4]). One can also show [5] that time-harmonic flows are, what remains when projecting certain diffeomorphism invariant Hamiltonian field theories on to the (integrable) Diff-singlet sector.

In this letter, I would like to show that time-harmonic flows may be deformed from the 'w_∞'-invariant parametrized form to 'W_∞'-invariant motions of parametrized surfaces, while keeping integrability in the form of infinitely many constants of motion. Again, the Lax-pair can be taken from the self-dual Yang-Mills theory, or rather: the Nahm equations (i.e. reduced self-dual $SU(N)$ Yang-Mills equations) are just a special case of the most natural first order (in time), quadratically non-linear, evolution equation for a set of (3) operators, X_i. The specification (of the space of operators) that will correspond to deformed time-harmonic flows of surfaces of spherical (toroidal) topology, resp. *-products on $S^2(T^2)$, is discussed in detail.

Let $\mathcal{A}$ be a non-commutative, associative Algebra ('of operators') and $X_i(t)_{i=1}^N$ a set of timedependent operators satisfying the non-linear evolution equation(s)

$$\dot{X}_i = \epsilon_{ii_1\ldots i_M} X_{i_1}, X_{i_2} \ldots X_{i_M} \tag{1}$$

$i = 1, 2, \ldots, N = M + 1$.

Using [6]/[7], one observes that (1) can be written in the form

$$\dot{L} = \Big[L, M_1, \ldots, M_{N-2}\Big], \tag{2}$$

where L and the M's ($\epsilon\mathcal{A}$) are linear combinations of the X_i, depending on $[\frac{N}{2}]$ spectral parameters, and $[\cdot, \ldots, \cdot]$ (a fully antisymmetric map from $\mathcal{A} \times \ldots \times \mathcal{A}$ to $\mathcal{A}$) denotes the natural M-commutator,

$$[A_1, \ldots, A_M] := \epsilon^{r_1\ldots r_M} A_{r_1} \cdot \ldots \cdot A_{r_M}. \tag{3}$$

In particular, one may take

$$L \;\; = \;\; \mu(X_1 + iX_2) - \Big(\frac{X_1 - iX_2}{\mu}\Big) + 2X_3$$

$$M_1 = i(\mu(X_1 + iX_2) + X_3) \tag{4}$$

(just as for the reduced self-dual Yang-Mills equations, see e.g. [2]) for $N = 3$, and (just as in [6]/[7])

$$\begin{aligned} L &= \mu(X_1 + iX_2) + \tilde{\mu}(X_3 + iX_4) \\ &- \frac{1}{\mu}(X_1 - iX_2) - \frac{1}{\tilde{\mu}}(X_3 - iX_4) + \sqrt{8}X_5 \\ M_1 &= \frac{i}{\sqrt{2}}\Big(\mu(X_1 + iX_2) + \frac{X_5}{\sqrt{2}}\Big) \\ M_2 &= \frac{i}{\sqrt{2}}\Big(\tilde{\mu}(X_3 + iX_4) + \frac{X_5}{\sqrt{2}}\Big) \\ M_3 &= \frac{-1}{\sqrt{2}}\Big(\frac{(X_1 - iX_2)}{\mu} - \frac{(X_3 - iX_4)}{\tilde{\mu}}\Big) \end{aligned} \tag{5}$$

for $N = 5$.

If there exists a trace on $\mathcal{A}$, satisfying

$$TrAB = TrBA, \tag{6}$$

$$Q_n := TrL^n, \quad n\epsilon\mathbf{N} \tag{7}$$

will be automatically time-independent only for $N = 3$; for odd $N > 3$, at least Q_1 and Q_2 are conserved, while for even N not even the basic M-commutator is traceless.

In the following, I will restrict myself to the case $N = 3$, i.e.

$$\dot{X}_i = \frac{1}{2}\epsilon_{ijk}[X_j, X_k] \tag{8}$$

which, if the X_i were finite-dimensional matrices, are just 'Nahm's equations'. They still trivially 'are', for infinite matrices with finitely many non-zero coefficients, but 'all' other infinite dimensional choices for $\mathcal{A}$ (or rather: an infinite-dimensional Lie-algebra, $\mathcal{L}$) are of quite different nature, and it seems that only the time-harmonic [1]/[5], w_∞-invariant case (area-preserving limit of $SU(N)$ [8],[3],[4]), where $\mathcal{L}$ is the Lie algebra of (non-constant) symplectic diffeomorphisms of S^2 or $T^2\ldots$, (8) becoming the following set of

first order partial differential equations for time-dependent functions on a two-dimensional manifold $\sum$,

$$\dot{x}_i = \frac{1}{2}\epsilon_{ijk}\frac{\epsilon^{rs}}{\rho(\phi)}\frac{\partial x^j}{\partial \phi^r}\frac{\partial x^k}{\partial \phi^s}, \tag{9}$$

has previously been considered in the literature ([3] - - [7]). Here, I would like to consider *-product deformations of (9), which amounts to (for S^2, [9]) choosing $\mathcal{A}$ to be the enveloping algebra of $SO(3)$, divided by the 'Casimir-ideal', or (for T^2) specific subclasses of infinite-dimensional matrices with only finitely many non-zero off-diagonals (cp. [10]). Both series of infinite dimensional 'W_∞'-algebras admit an invariant trace, making (7) time-independent for all n.

Let me first discuss the 'spherical type' W^∞-algebras (cp. [9], [11]).

Let $\mathcal{G}$ be a semi-simple Lie-algebra, $\{T_a\}_{a=1}^{d=dim\mathcal{G}}$ a basis of $\mathcal{G}$,

$$[T_a, T_b] = f_{ab}^c T_c \qquad abc = 1\dots d \tag{10}$$

and $U(\mathcal{G})$ be the associative algebra (over $\mathbf{C}$) of polynomials

$$T = c_T \cdot \mathbf{1} + \sum c^{a_1\dots a_2} T_{a_1}\dots T_{a_l}, \tag{11}$$

modulo (10) (i.e. the universal enveloping algebra). The center of U is generated by $r = \text{rank}\mathcal{G}$ 'Casimirs' $C_1, \dots, C_r$, and U may be divided by the sum of the r two-sided ideals

$$I_j = (C_j - \lambda_j \mathbf{1})U, \qquad \lambda_j \epsilon \mathbf{C}; \tag{12}$$

resulting in $U_{\lambda=(\lambda_1,\dots,\lambda_r)}(\mathcal{G})$, the algebra of polynomials

$$T^{(\lambda)} = c_T\mathbf{1} + \sum c_T^{a_1\dots a_l} T_{a_1}^{(\lambda)} \cdot \dots \cdot T_{a_l}^{(\lambda)}, \tag{13}$$

where the $T_a^{(\lambda)}$ are irreducible representations of (10), having the property that certain polynomials, like

$$C_1^{(\lambda)} := T_a T_b g^{ab} = \lambda_1 \cdot \mathbf{1} \tag{14}$$

(g^{ab} being the inverse of $g_{ab} := \frac{1}{2} f_{ac}^d f_{bd}^c$, the metric tensor on $\mathcal{G}$) are proportional to $\mathbf{1}$, and the coefficients $c^{a_1\dots a_l}$ apart from (for definiteness)

being totally antisymmetric, will consequently be taken to satisfy additional requirements like $g_{ab}c^{aba_3\dots a_l} \equiv 0, \dots$ (in accordance with the r Casimir relations). It is known (see e.g. [12]) that $U_\lambda(\mathcal{G})$ decomposes, under the action of $\mathcal{G}$, into a direct sum of finite dimensional irreducible $\mathcal{G}$-moduls,

$$U_\lambda = \oplus \sum_{(t)_j} U_\lambda^{(t)_j}, \tag{15}$$

$j = 1 \dots m(t)$, where each (tensor) representation (t) occurs finitely many $(m_{(t)})$ times; the 1-dimensional representation occurs only once and, as $[U_\lambda, U_\lambda] = [\mathcal{G}_\lambda, U_\lambda]$ (see [13]), it is easy to see that U_λ is also the direct sum

$$U_\lambda = \mathbf{C} \cdot \mathbf{1} \oplus [U_\lambda, U_\lambda], \tag{16}$$

implying that

$$TrT := c_T \tag{17}$$

defines an invariant trace on U_λ, $Tr[A, B] = 0$ – which is all one needs to conclude that (8), with $\mathcal{A} = U_\lambda(\mathcal{G})$, will have infinitely many conserved quantities, (7). I am referring to the series of algebras $U_\lambda(\mathcal{G})$ as the 'spherical series' as in the simplest case, $\mathcal{G} = SO(3)$, the (to be extended) map $\phi_{\hbar := \frac{1}{\lambda}}$,

$$\begin{aligned} Y_{lm}(\theta, \phi) \quad &= \quad (\sum c^{(lm)}_{a_1 \dots a_l} x_{a_1} \cdot \dots \cdot x_{a_l})_{\vec{x} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)} \\ &\longrightarrow \quad \phi_\hbar(Y_{lm}) = \bar{T}^{(\lambda)}_{lm} := c^{(\lambda)}_{(l)} \sum c^{(lm)}_{a_1 \dots a_l} \bar{T}^{(\lambda)}_{a_1 \dots} \bar{T}^{(\lambda)}_{a_l}, \end{aligned} \tag{18}$$

where $[\bar{T}^{(\lambda)}_a, \bar{T}^{(\lambda)}_b] = \frac{\epsilon_{abc}}{\lambda} \bar{T}^{(\lambda)}_c, \sum_{a=1}^3 \bar{T}^{(\lambda)}_a \bar{T}^{(\lambda)}_a = \mathbf{1}$, provides a one to one correspondence between $U_\lambda(SO(3))$ and the (commutative, resp. Poisson-) algebra of complex-valued functions on S^2 (cp. [8], [14]). Moreover, the overall normalisation may be chosen such that when $\hbar = \frac{1}{\lambda}$ is appropriately taken to 0,

$$\begin{aligned} \frac{1}{i\hbar}[\phi_\hbar(Y_{lm}), \phi_\hbar(Y_{l'm'})] \quad &\to \quad \{Y_{lm}, Y_{l'm'}\} \\ &:= \quad \frac{1}{sin\theta}(\frac{\partial Y_{lm}}{\partial \theta}\frac{\partial Y_{l'm'}}{\partial \phi} - (lm \longleftrightarrow l'm')) \end{aligned} \tag{19}$$

- – which together with suitable properties under complex (hermitean) conjugation, ... (see [15] for the definition of a star-product) allows to call the associative multiplication in $U_\lambda(SO(3))$ a *-product on S^2:

$$Y_{lm} * Y_{l'm'} := \phi_\hbar^{-1}(\phi_\hbar(Y_{lm})\phi_\hbar(Y_{l'm'})). \tag{20}$$

For the 'Torus-case', where a *-product may easily be written directly in terms of functions on T^2,

$$f * g := f \cdot g + \sum_{n=1}^{\infty} \frac{(\lambda/2)^n}{n!} \epsilon^{r_1 s_1} \ldots \epsilon^{r_n s_n} \frac{\partial^n f}{\partial\phi^{r_1} \ldots \partial\phi^{r_n}} \frac{\partial^n g}{\partial\phi^{s_1} \ldots \partial\phi^{s_n}}, \tag{21}$$

the equation

$$\lambda \dot{x}_i = \epsilon_{ijk} x_j * x_k, \tag{22}$$

viewed as an evolution-equation for a hypersurface in $\mathbf{R}^3$, may then 'at any stage' (s.b.) be compared with the time-harmonic flow (9) (which is equaivalent to (22), $\lambda = 0$). In view of (9) being equivalent to (cp. [1])

$$\vec{\nabla}^2 t(x^1, x^2, x^3) = 0, \tag{23}$$

it is tempting to interchange dependent and independent variables also in (22): making this 'hodograph' transformation,

$$\phi^o = t, \phi^1, \phi^2 \to x^i := x^i(t, \phi^1, \phi^2), \tag{24}$$

one first notes the purely 'kinematical' consequences,

$$\begin{aligned}
\dot{\vec{x}} &\hat{=} J(\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2) \\
J : &= |(\frac{\partial x^i}{\partial \phi^\alpha})| = \dot{\vec{x}}(\partial_1 \vec{x} \times \partial_2 \vec{x}) = (\vec{\nabla} t \cdot (\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2))^{-1} \\
\partial_t &\hat{=} \dot{\vec{x}} \cdot \vec{\nabla} = J(\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2) \cdot \vec{\nabla} =: D_o \\
\frac{\partial}{\partial \phi^r} &\hat{=} J(\vec{\nabla} t \times \vec{\nabla}\phi^r) \cdot \vec{\nabla} =: D^r \\
D^r \phi^s &= \epsilon^{rs}, \quad [D^\alpha, D^\beta] = 0,
\end{aligned} \tag{25}$$

$\alpha, \beta = 0, 1, 2$

while (22) becomes

$$\begin{aligned}
(\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2)_i &= \frac{\epsilon_{ijk}}{\lambda J} \quad e^{\frac{\lambda}{2}\epsilon_{rs} D^r \otimes D^s} x_j \otimes x_k \Big|_{\text{Diag.}} \\
&= (\vec{\nabla} t)_i \\
&+ \frac{1}{2J} \epsilon_{ijk} \sum_{l=1}^{\infty} \frac{(\lambda^2/4)^l}{(2l+1)!} (\epsilon_{rs} D^r \otimes D^s)^{2l+1} x_j \otimes x_k \Big|_{\text{Diag.}} \\
&=: (\vec{\nabla} t)_i + \sum_{l=1}^{\infty} (\lambda^2)^l (\vec{F}_l(\vec{\nabla} t, \vec{\nabla}\phi^1, \vec{\nabla}\phi^2))_i.
\end{aligned} \tag{26}$$

Note that just as $\sum_i [x_i, \dot{x}_i]_* = 0$ is a consequence of (22), solutions of (26) will satisfy

$$\begin{aligned}\sum_i e^{\frac{\lambda}{2}\epsilon_{rs}D^r\otimes D^s} & \left(x_i \otimes J(\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2)_i - J(\vec{\nabla}\phi^1 \times \vec{\nabla}\phi^2)_i \otimes x_i\right)\Big|_{\text{Diag.}} \\ & = 0 \end{aligned} \tag{27}$$

At least recursively, (26) is still solvable, as expanding the 3 unknown functions t, ϕ^1, ϕ^2 into powerseries in λ^2,

$$\begin{aligned} t(\vec{x}) &= T(\vec{x}) + \sum_{n=1}^{\infty} \lambda^{2n} t_n(\vec{x}) \\ \phi^r(\vec{x}) &= \phi^r(\vec{x}) + \sum_{n=1}^{\infty} \lambda^{2n} \phi_n^r(\vec{x}), \end{aligned} \tag{28}$$

the zero'th order (non-linear) ones <u>are</u> solvable, while all the higher ones are (recursively) linear; in particular, all $t_n(\vec{x})$ are given as solutions of Poisson's equation,

$$\vec{\nabla}^2 t_n(\vec{x}) = \vec{\nabla} G_n, \tag{29}$$

with G_n only depending on the $t_{m<n}$, $\phi^1_{m<n}$ and $\phi^2_{m<n}$. Of course, it would be desirable to derive (from (26)) an equation only involving t.

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